PHYS 705: Classical Mechanics

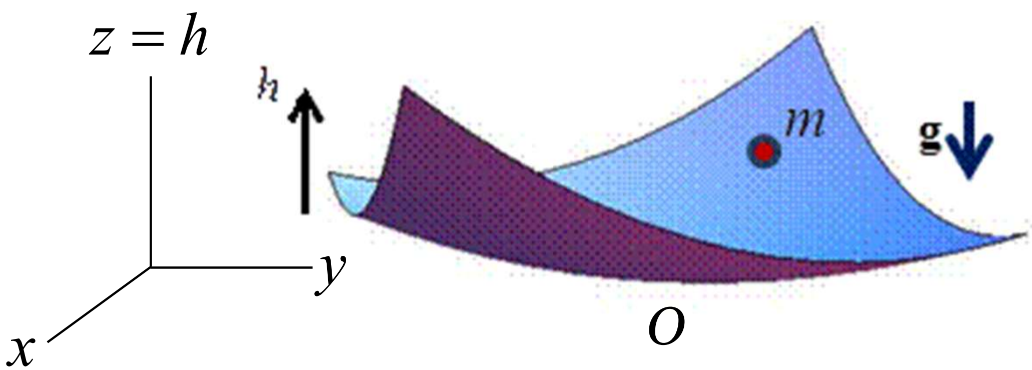
Housekeeping

- Today is our last lecture

  Thank you for a good semester!

- Final Exam next week on Dec. 6

Additional Problem (HW 10)



$$h(x, y) = x^2 + y^2 - xy$$

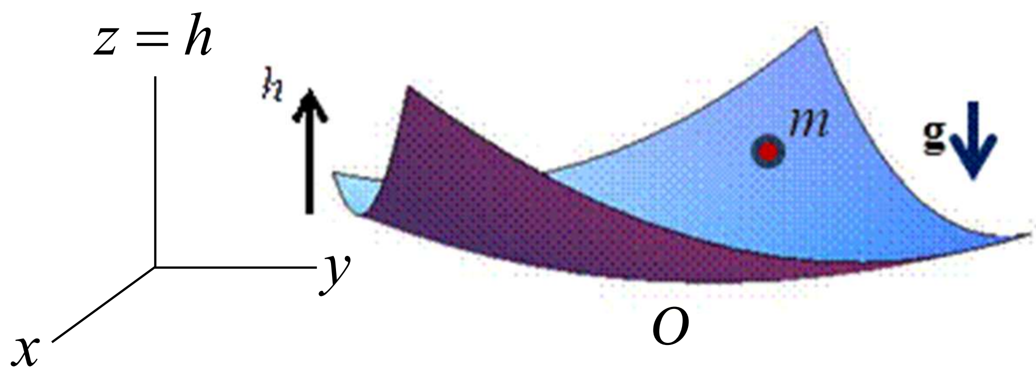
Choose origin to be at bottom of bowl

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad V = mg\left(x^2 + y^2 - xy\right)$$

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mg\left(x^2 + y^2 - xy\right)$$

Equilibrium @ $(x_0, y_0, z_0) = (0,0,0)$, $(x, y, z) = (\eta_x, \eta_y, \eta_z) \approx (0,0,0)$

Additional Problem (HW 10)



$$h(x,y) = x^2 + y^2 - xy$$

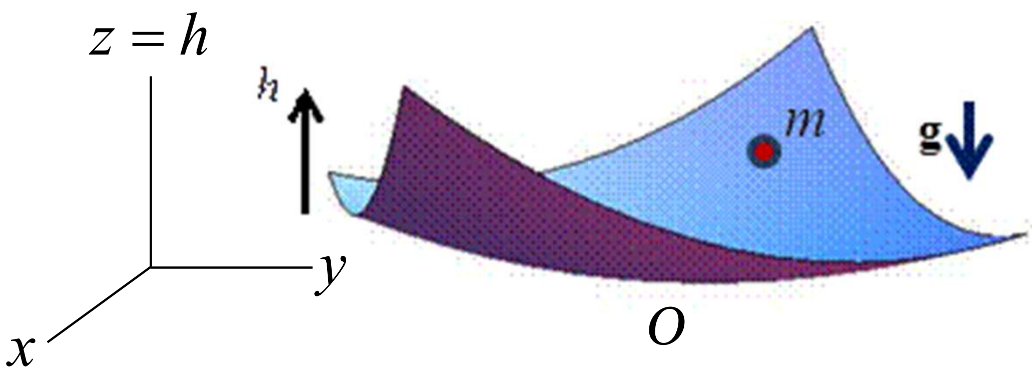
Since $z = h(x,y) = x^2 + y^2 - xy$, we have $\dot{z} = (2x - y)\dot{x} + (2y - x)\dot{y}$

and $\dot{z}^2 = (2x-y)^2\dot{x}^2 + (2y-x)^2\dot{y}^2 + 2(2x-y)(2y-x)\dot{x}\dot{y}$

Near $(x_0, y_0) = (0,0)$, keeping only up to quadratic terms, we have:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad V = mg\left(x^2 + y^2 - xy\right)$$

Additional Problem (HW 10)



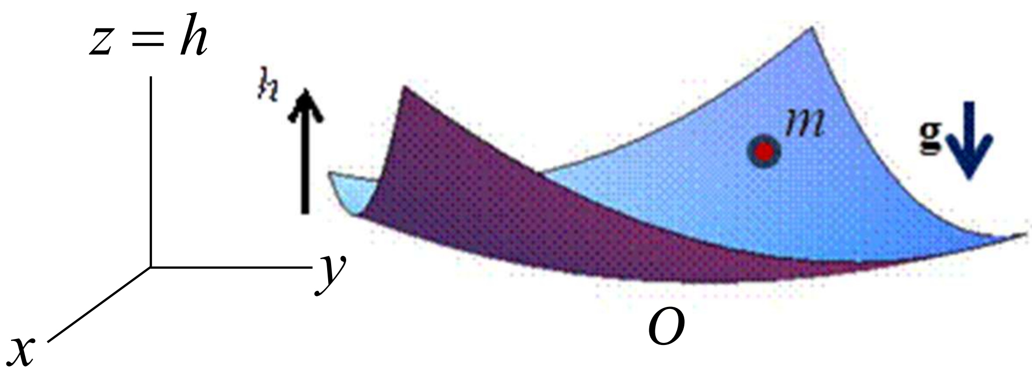
$$h(x,y) = x^2 + y^2 - xy$$

So, near our equilibrium $(x_0, y_0) = (0,0)$,

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) \quad \Rightarrow \quad T_{ij} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$V = mg\left(x^2 + y^2 - xy\right) \quad \Rightarrow \quad V_{ij} = \begin{pmatrix} 2mg & -mg \\ -mg & 2mg \end{pmatrix}$$

Additional Problem (HW 10)



$$h(x, y) = x^2 + y^2 - xy$$

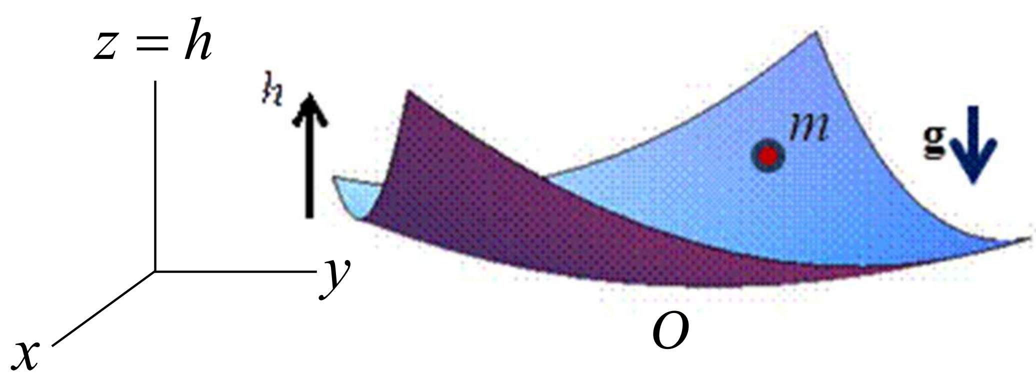
The Characteristic equation for the eigenvalues is:

$$\det\left(\mathbf{V} - \omega^2\mathbf{T}\right) = m\left[\left(2g - \omega^2\right)^2 - g^2\right] = 0$$

$$\left(2g - \omega^2 - g\right)\left(2g - \omega^2 + g\right) = 0$$

$$\left(g - \omega^2\right)\left(3g - \omega^2\right) = 0$$

Additional Problem (HW 10)



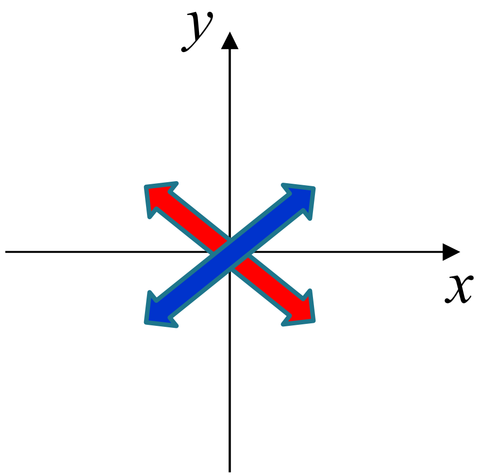
$$h(x,y) = x^2 + y^2 - xy$$

$$\omega_+ = \sqrt{3g}$$
$$\omega_- = \sqrt{g}$$

with

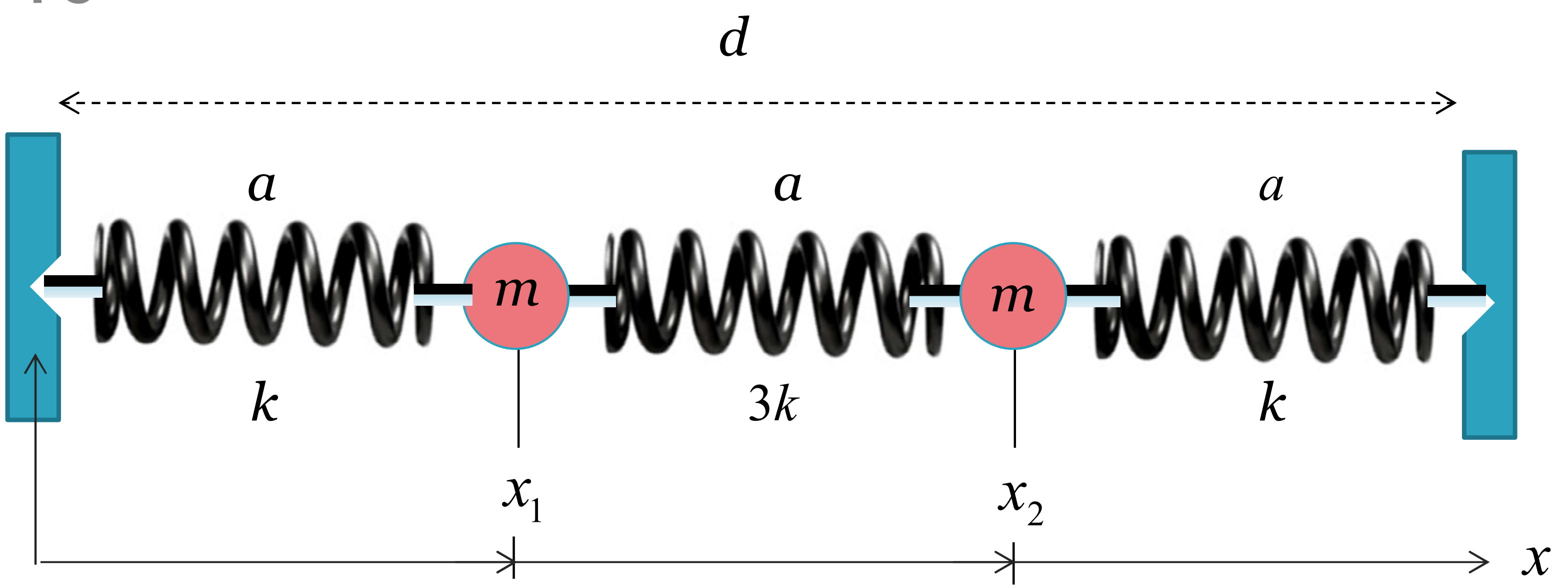
$$\mathbf{a}_+ = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\mathbf{a}_- = \frac{1}{\sqrt{2m}}\begin{pmatrix}1\\1\end{pmatrix}$$



$$\mathbf{a}_\pm^T \mathbf{T} \mathbf{a}_\pm = 1$$

HW #10 Prob 6.12



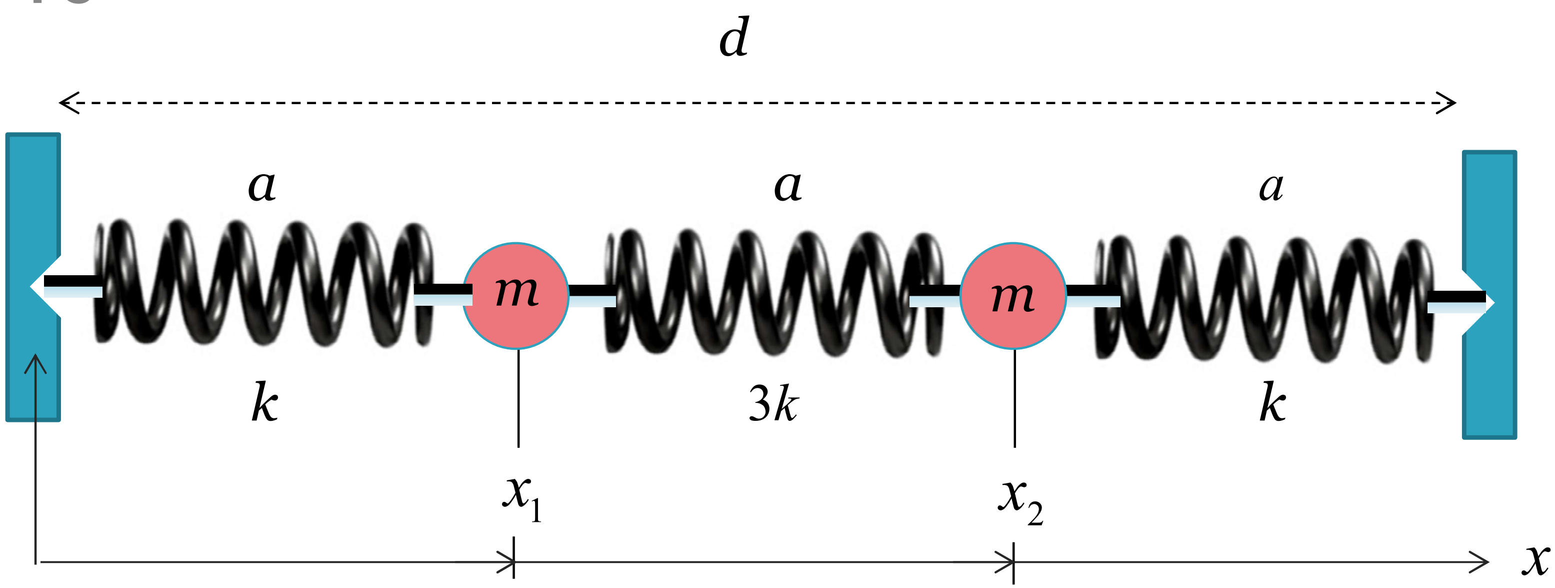
- a is the **unstretched** equilibrium length of the springs.

NOTE: if the total length of the system d is not = 3a,

- Then, the equilibrium positions for the masses are NOT necessarily at

$$x_{10} = a \qquad x_{20} = 2a$$

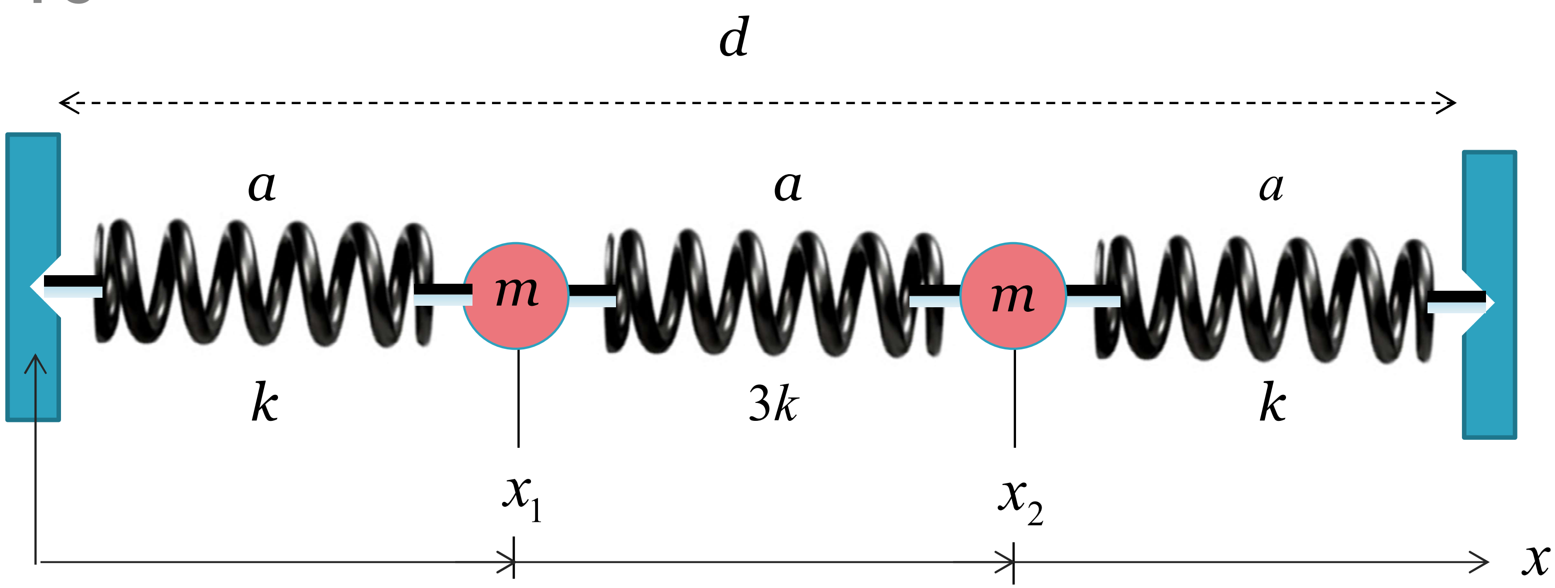
HW #10 Prob 6.12



Using the left fixed end as a reference point, x_j gives the instantaneous positions of m_j.

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) \qquad U = \frac{k}{2}\left(x_1 - a\right)^2 + \frac{3k}{2}\left(\left(x_2 - x_1\right) - a\right)^2 + \frac{k}{2}\left(\left(d - x_2\right) - a\right)^2$$

HW #10 Prob 6.12



The equilibrium positions x_{10} and x_{20} are given by,

$$\frac{\partial U}{\partial x_1} = k(x_1 - a) - 3k(x_2 - x_1 - a) = 0$$

AND

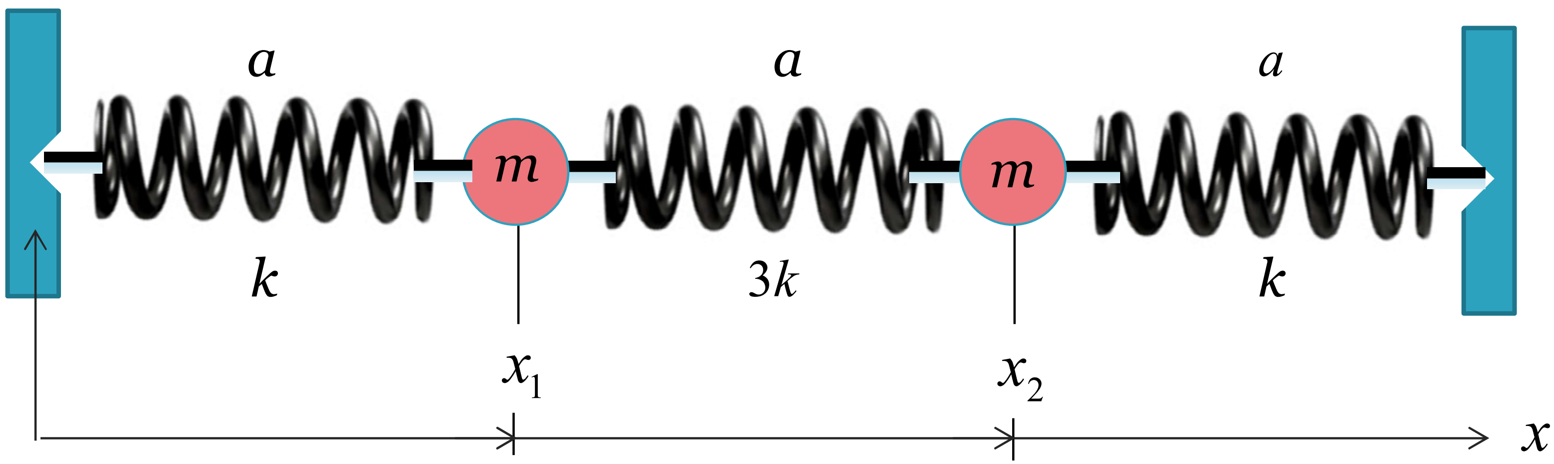
$$\frac{\partial U}{\partial x_2} = 3k(x_2 - x_1 - a) - k(d - x_2 - a) = 0$$

$\Rightarrow$

$$x_{10} = \frac{3d - 2a}{7}$$

$$x_{20} = \frac{4d + 2a}{7}$$

HW #10 Prob 6.12



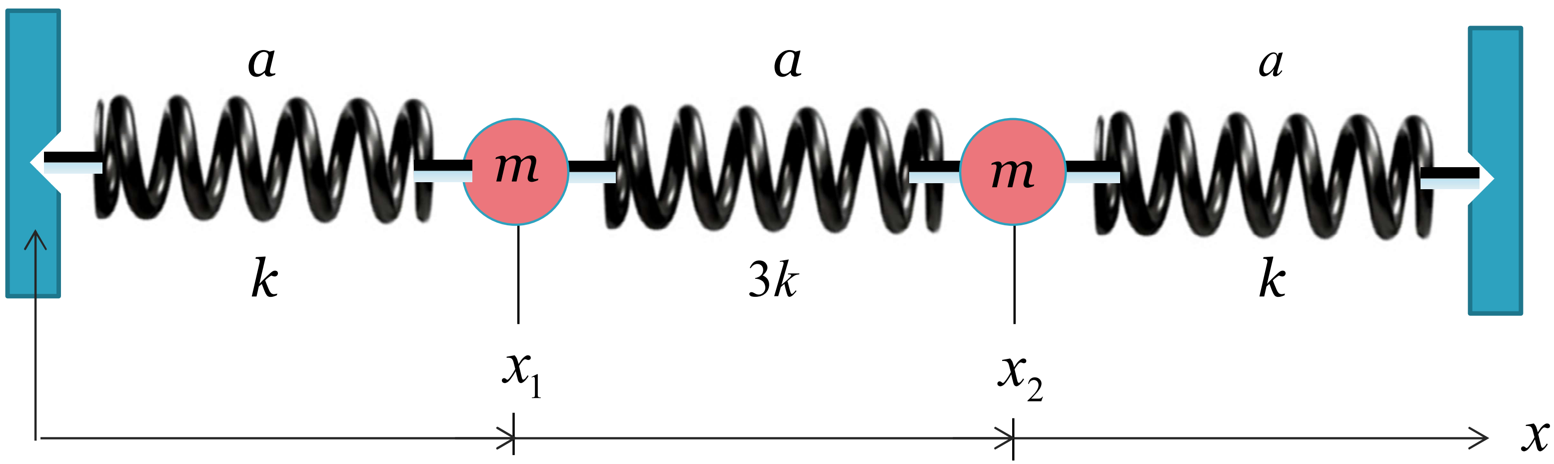
The deviations (η_1, η_2) are defined wrt to these two eq values: x_{10}, x_{20}

$$\eta_j = x_j - x_{j0}$$

T_{ij} is easy to calculate since $\dot{\eta}_j = \dot{x}_j$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) \Rightarrow T = \frac{m}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2\right) \Rightarrow T_{ij} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

HW #10 Prob 6.12

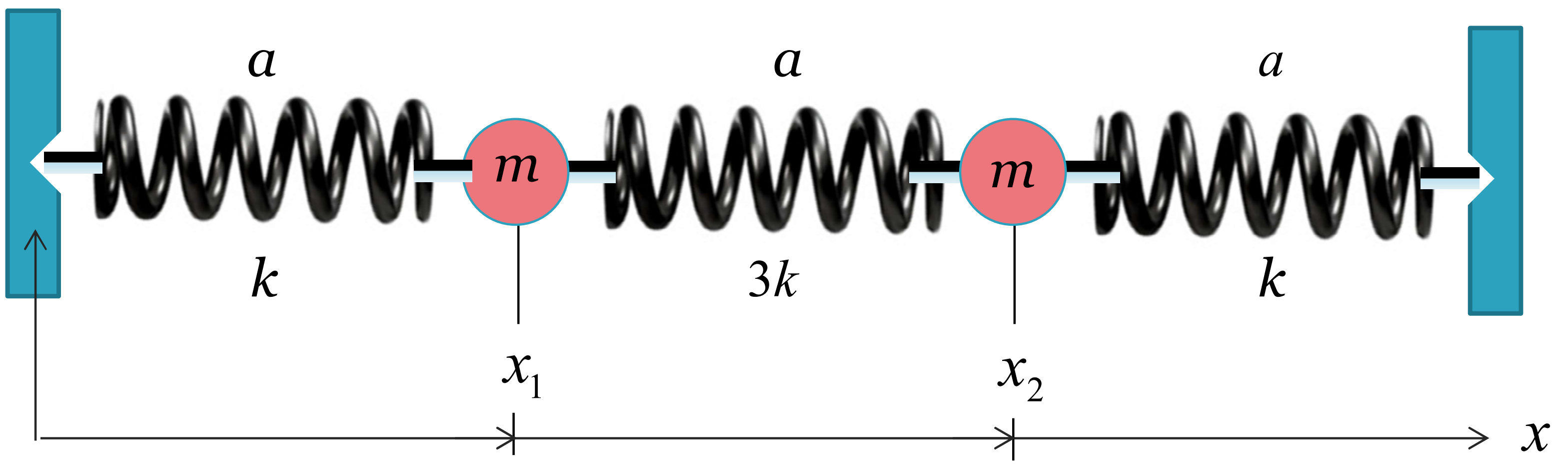


For U_{ij} , since U is a quadratic function, one can expand U out and pick out all the quadratic terms

$$U = \frac{k}{2}(x_1 - a)^2 + \frac{3k}{2}(x_2 - x_1 - a)^2 + \frac{k}{2}(d - x_2 - a)^2$$

$$U = \frac{k}{2}(\eta_1 + x_{10} - a)^2 + \frac{3k}{2}(\eta_2 + x_{20} - \eta_1 - x_{10} - a)^2 + \frac{k}{2}(d - \eta_2 - x_{20} - a)^2$$

HW #10 Prob 6.12



$$U = \frac{k}{2}\left(\eta_1 + x_{10} - a\right)^2 + \frac{3k}{2}\left(\eta_2 + x_{20} - \eta_1 - x_{10} - a\right)^2 + \frac{k}{2}\left(d - \eta_2 - x_{20} - a\right)^2$$

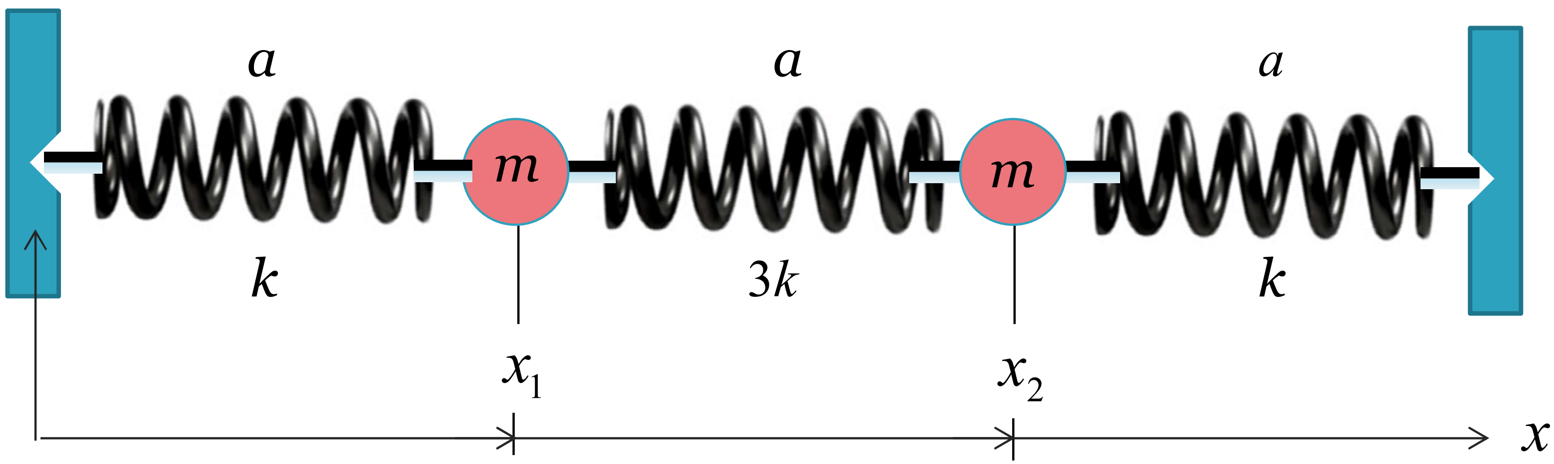
Near the equilibrium values $\eta_j \simeq 0$ and substitute values for x_{10}, x_{20},

U_{ij} is then picked out simply as the quadratic terms:

$$x_{10} = \frac{3d - 2a}{7} = a \qquad x_{20} = \frac{4d + 2a}{7} = 2a$$

$$U\big|_{x_0} = \frac{k}{2}\left(\eta_1 \cancel{+a} \cancel{-a}\right)^2 + \frac{3k}{2}\left(\eta_2 \cancel{+2a} - \eta_1 \cancel{-a - a}\right)^2 + \frac{k}{2}\left(\cancel{3a} - \eta_2 \cancel{-2a - a}\right)^2$$

HW #10 Prob 6.12

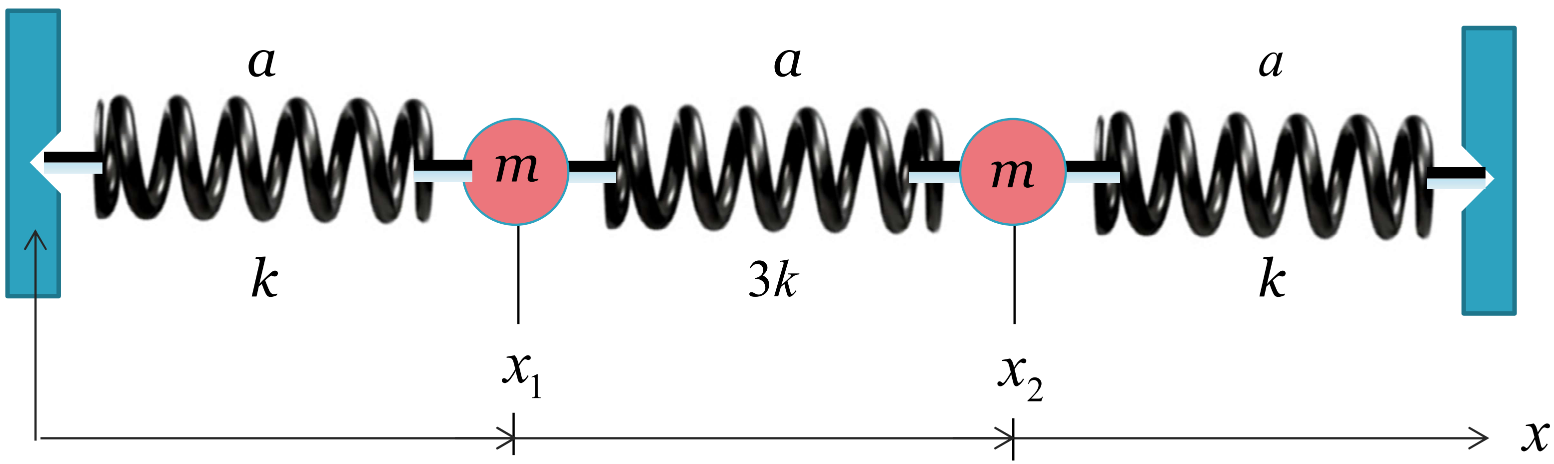


$$U = \frac{k}{2}\left(\eta_1 + x_{10} - a\right)^2 + \frac{3k}{2}\left(\eta_2 + x_{20} - \eta_1 - x_{10} - a\right)^2 + \frac{k}{2}\left(d - \eta_2 - x_{20} - a\right)^2$$

$$U\Big|_{x_0} = \frac{k}{2}\eta_1^{\ 2} + \frac{3k}{2}\left(\eta_2^2 - 2\eta_1\eta_2 + \eta_1^2\right) + \frac{k}{2}\eta_2^2$$

$$\Rightarrow \quad U_{ij} = \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix}$$

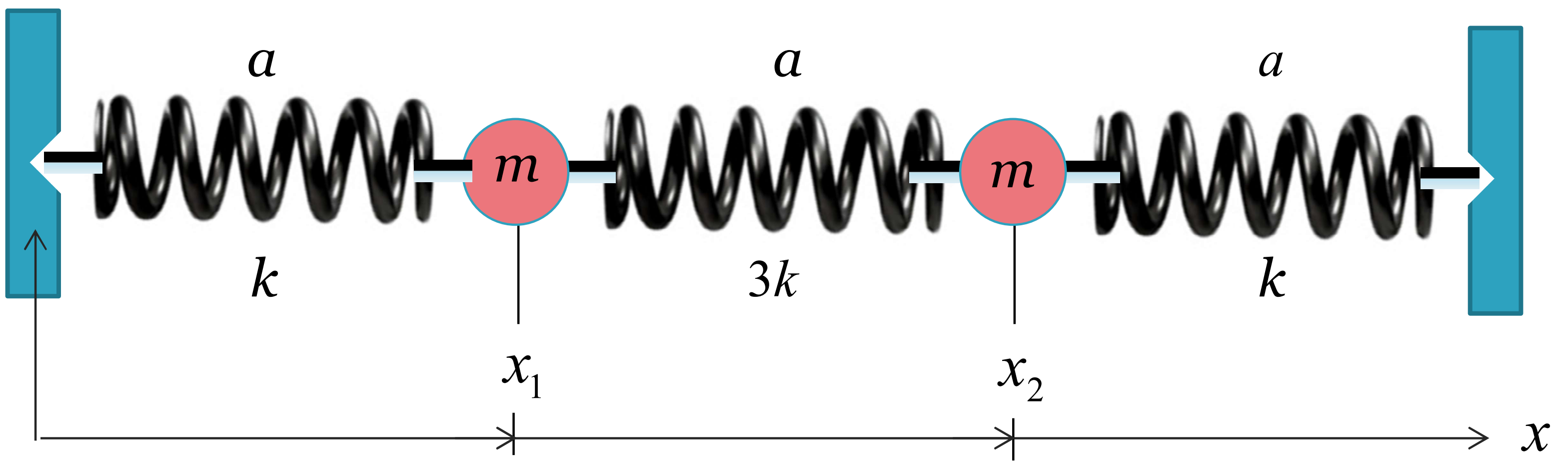
HW #10 Prob 6.12



Alternative, the quadratic form U_{ij} is the approximation of U about x_{10}, x_{20}

$$U_{ij} = \left. \frac{\partial^2 U}{\partial x_i \partial x_j} \right|_{x_0}$$

HW #10 Prob 6.12



The simplest way to get U_{ij} is to directly evaluate these double derivatives:

$$\frac{\partial U}{\partial x_1} = k(x_1 - a) - 3k(x_2 - x_1 - a) = 0$$

$$\frac{\partial U}{\partial x_2} = 3k(x_2 - x_1 - a) - k(d - x_2 - a) = 0$$

$$U_{11} = k + 3k = 4k$$

$$U_{12} = U_{21} = -3k$$

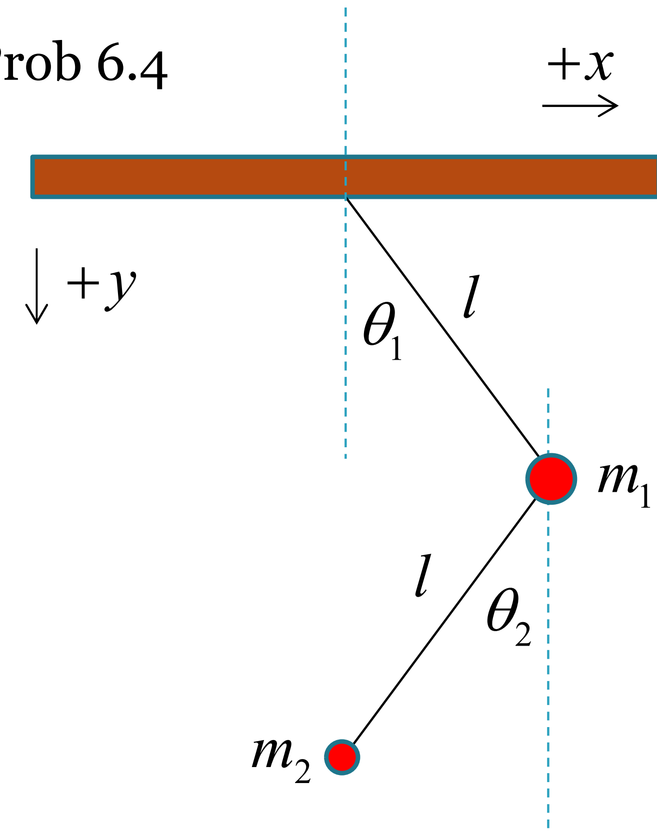
$$U_{22} = 4k$$

$$U_{ij} = \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix}$$

taking derivative one more time $U_{ij} = \left.\frac{\partial^2 U}{\partial x_i \partial x_j}\right|_{x_0}$

HW #10

Prob 6.4



$$x_1 = l\sin\theta_1 \qquad x_2 = l\sin\theta_1 - l\sin\theta_2$$

$$y_1 = l\cos\theta_1 \qquad y_2 = l\cos\theta_1 + l\cos\theta_2$$

$$\dot{x}_1 = l\cos\theta_1\dot{\theta} \qquad \dot{x}_2 = l\cos\theta_1\dot{\theta}_1 - l\cos\theta_2\dot{\theta}_2$$

$$\dot{y}_1 = -l\sin\theta_1\dot{\theta} \qquad \dot{y}_2 = -l\sin\theta_1\dot{\theta}_1 - l\sin\theta_2\dot{\theta}_2$$

$$T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{m_1}{2}l^2\dot{\theta}_1^2 + \frac{m_2}{2}\left(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 - 2l^2\cos\left(\theta_1 + \theta_2\right)\dot{\theta}_1\dot{\theta}_2\right)$$

HW #10

$$x_1 = l\sin\theta_1 \qquad x_2 = l\sin\theta_1 - l\sin\theta_2$$
$$y_1 = l\cos\theta_1 \qquad y_2 = l\cos\theta_1 + l\cos\theta_2$$

$$U = m_1 g\left(l - y_1\right) + m_2 g\left(2l - y_1 - y_2\right)$$
$$= m_1 gl\left(1 - \cos\theta_1\right) + m_2 gl\left(2 - \cos\theta_1 - \cos\theta_2\right)$$

$$U = 0 \ @\ \theta_1 = \theta_2 = 0$$

$$L = T - U = \frac{m_1}{2} l^2\dot{\theta}_1^2 + \frac{m_2}{2}\left(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 - 2l^2\cos\left(\theta_1 + \theta_2\right)\dot{\theta}_1\dot{\theta}_2\right)$$
$$- m_1 gl\left(1 - \cos\theta_1\right) - m_2 gl\left(2 - \cos\theta_1 - \cos\theta_2\right)$$

Letting $\eta_j = \theta_j - \theta_{0j}$ near the equilibrium at $\theta_{01} = \theta_{02} = 0$, we have:

$$\eta_j = \theta_j \approx 0 \qquad 1 - \cos\theta_j \simeq \theta_j^2/2 = \eta_j^2/2$$

$$\dot{\eta}_j = \dot{\theta}_j$$

HW #10

Note: To the lowest order in η_j, we also have:

$$\cos(\theta_1+\theta_2)\dot{\theta}_1\dot{\theta}_2 = \left[1+O\left(\eta^2\right)\right]\dot{\eta}_1\dot{\eta}_2 \simeq \dot{\eta}_1\dot{\eta}_2$$

Thus, approximating T around the equilibrium, we have,

$$T = \frac{m_1}{2} l^2\dot{\theta}_1^2 + \frac{m_2}{2}\left(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 - 2l^2\cos(\theta_1+\theta_2)\dot{\theta}_1\dot{\theta}_2\right)$$

$$\Downarrow$$

$$T \simeq \frac{1}{2} l^2(m_1+m_2)\dot{\eta}_1^2 + \frac{1}{2} l^2 m_2\dot{\eta}_2^2 - l^2 m_2\dot{\eta}_1\dot{\eta}_2$$

$$L \simeq \frac{1}{2} T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2} U_{ij}\eta_i\eta_j$$

$$\Downarrow$$

$$T_{ij} = l^2\begin{pmatrix} m_1+m_2 & -m_2 \\ -m_2 & m_2 \end{pmatrix}$$

HW #10

For $\theta_j = \eta_j \simeq 0$, recall we have $1 - \cos\theta_j \simeq \theta_j^2/2 = \eta_j^2/2$

Thus, approximating U around equilibrium, we have,

$$U = m_1 gl(1 - \cos\theta_1) + m_2 gl(1 - \cos\theta_1 + 1 - \cos\theta_2)$$

$$U \simeq \frac{1}{2}\left[(m_1 + m_2) gl\eta_1^2 + m_2 gl\eta_2^2\right]$$

$$L \simeq \frac{1}{2} T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2} U_{ij}\eta_i\eta_j$$

$$U_{ij} = gl\begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

HW #10

Resonant frequencies (eigen-frequencies) are given by the solution of the characteristic equation:

$$\det\left(U_{ij} - \lambda T_{ij}\right) = 0 \quad \Rightarrow \quad \det\begin{pmatrix} \left(gl - l^2\lambda\right)\left(m_1 + m_2\right) & l^2\lambda m_2 \\ l^2\lambda m_2 & \left(gl - l^2\lambda\right)m_2 \end{pmatrix} = 0$$

$$\left(gl - l^2\lambda\right)^2 \left(m_1 + m_2\right)m_2 - \left(l^2 m_2^2 \lambda\right)^2 = 0$$

$$\left[\left(gl - l^2\lambda\right)\sqrt{} - l^2 m_2^2 \lambda\right]\left[\left(gl - l^2\lambda\right)\sqrt{} + l^2 m_2^2 \lambda\right] = 0$$

with $\sqrt{} = \sqrt{\left(m_1 + m_2\right)m_2}$

HW #10

$$\left[\left(gl-l^2\lambda\right)\sqrt{\quad}+l^2m_2^2\lambda\right]\left[\left(gl-l^2\lambda\right)\sqrt{\quad}-l^2m_2^2\lambda\right]=0$$

This equation has two solutions:

$$\left(gl-l^2\lambda_+\right)\sqrt{\quad}=-l^2m_2^2\lambda_+ \qquad\qquad \left(gl-l^2\lambda_-\right)\sqrt{\quad}=l^2m_2^2\lambda_-$$

⇓ ⇓

$$\lambda_+=\frac{g}{l}\frac{\sqrt{m_2\left(m_1+m_2\right)}}{\sqrt{m_2\left(m_1+m_2\right)}-m_2} \qquad\qquad \lambda_-=\frac{g}{l}\frac{\sqrt{m_2\left(m_1+m_2\right)}}{\sqrt{m_2\left(m_1+m_2\right)}+m_2}$$

The resonant frequencies are given by $\omega=\sqrt{\lambda}$:

$$\omega_\pm=\sqrt{\frac{g}{l}}\left(\frac{\sqrt{m_2\left(m_1+m_2\right)}}{\sqrt{m_2\left(m_1+m_2\right)}\mp m_2}\right)^{1/2}$$

HW #10

$$\omega_{\pm} = \sqrt{\frac{g}{l}}\left(\frac{\sqrt{m_2(m_1+m_2)}}{\sqrt{m_2(m_1+m_2)}\mp m_2}\right)^{1/2}$$

Defining the following,

$$M = m_1 + m_2 \gg m_2 \qquad \varepsilon = m_2/M \ll 1 \qquad (m_2 \ll m_1)$$

We can write:

$$\left(m_2(m_1+m_2)\right)^{1/2} = \left(M^2\frac{m_2}{M}\right)^{1/2} = M\sqrt{\varepsilon}$$

note: this is still exact.

So, the resonant frequencies can be written as:

$$\omega_{\pm} = \sqrt{\frac{g}{l}}\left(\frac{M\sqrt{\varepsilon}}{M\sqrt{\varepsilon}\mp m_2}\right)^{1/2} = \sqrt{\frac{g}{l}}\left(\frac{\not{M}\sqrt{\varepsilon}}{\not{M}\left(\sqrt{\varepsilon}\mp\varepsilon\right)}\right)^{1/2} = \sqrt{\frac{g}{l}}\left(\frac{1}{1\mp\sqrt{\varepsilon}}\right)^{1/2}$$

$(m_2 \ll m_1)$ ⇒

$$\omega_{\pm} \simeq \sqrt{\frac{g}{l}}\left(1 \pm \frac{\sqrt{\varepsilon}}{2}\right)$$

⇒

$$\omega_{\pm} \simeq \sqrt{\frac{g}{l}}, \quad \varepsilon \to 0$$

HW #10

Now, the associated eigenvectors can be calculated from,

$$\left(\mathbf{U}-\omega_{\pm}^{2}\mathbf{T}\right)\cdot\mathbf{a}_{\pm}=\mathbf{0}$$

$$\omega_{+}\simeq\sqrt{\frac{g}{l}}\left(1+\frac{\sqrt{\varepsilon}}{2}\right)\quad\Rightarrow\quad a_{+}^{(1)}=-\sqrt{\varepsilon}a_{+}^{(2)}$$

$$\omega_{-}\simeq\sqrt{\frac{g}{l}}\left(1-\frac{\sqrt{\varepsilon}}{2}\right)\quad\Rightarrow\quad a_{-}^{(1)}=\sqrt{\varepsilon}a_{-}^{(2)}$$

Normalization wrt to $\tilde{\mathbf{a}}_{\pm}\cdot\mathbf{T}\cdot\mathbf{a}_{\pm}=\mathbf{1}$ gives:

$$\mathbf{a}_{+}=N_{+}\begin{pmatrix}-\sqrt{\varepsilon}\\1\end{pmatrix}\quad\text{and}\quad\mathbf{a}_{-}=N_{-}\begin{pmatrix}\sqrt{\varepsilon}\\1\end{pmatrix}\qquad N_{\pm}=\left[l^{2}M\left(\varepsilon^{2}\pm\varepsilon^{3/2}+1\right)\right]^{-1/2}$$

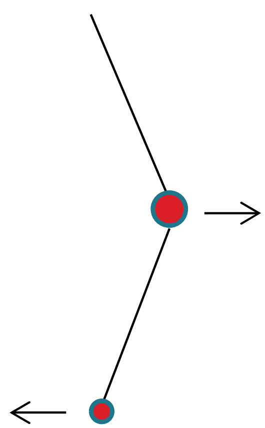
HW #10

Recall that the generalized coordinates and the **normal modes** ζ are related by:

$$\boldsymbol{\eta} = \mathbf{A} \cdot \boldsymbol{\zeta} \quad \text{where} \quad \zeta = \begin{pmatrix} C_+ e^{i\omega_+ t} \\ C_- e^{i\omega_- t} \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} a_+^{(1)} & a_-^{(1)} \\ a_+^{(2)} & a_-^{(2)} \end{pmatrix} = \begin{pmatrix} -\sqrt{\varepsilon} N_+ & \sqrt{\varepsilon} N_- \\ N_+ & N_- \end{pmatrix}$$

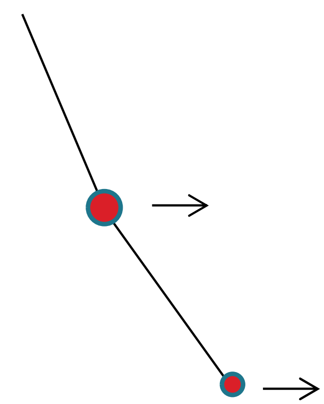
Only ζ_+ is active (anti-symmetric)

$$\eta_1 \sim -\sqrt{\varepsilon}\zeta_+$$
$$\eta_2 \sim \zeta_+$$



Only ζ_- is active (symmetric)

$$\eta_1 \sim +\sqrt{\varepsilon}\zeta_-$$
$$\eta_2 \sim \zeta_-$$



HW #10

We just saw that the general solution can be written as a linear combination of the **normal modes**:

$$\omega_\pm \simeq \sqrt{\frac{g}{l}}\left(1 \pm \frac{\sqrt{\varepsilon}}{2}\right)$$

$$\bar{\eta}_1(t) = \mathrm{Re}\left[C_+ a_+^{(1)} e^{i\omega_+ t} + C_- a_-^{(1)} e^{i\omega_- t}\right]$$

$$\bar{\eta}_2(t) = \mathrm{Re}\left[C_+ a_+^{(2)} e^{i\omega_+ t} + C_- a_-^{(2)} e^{i\omega_- t}\right]$$

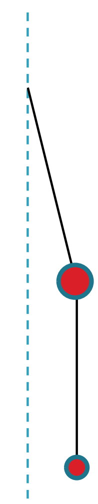
$$\mathbf{a}_\pm = N_\pm \begin{pmatrix} \mp\sqrt{\varepsilon} \\ 1 \end{pmatrix}$$

The constants C_+ and C_- will be determined by initial conditions.

With the prescribed pluck: $\bar{\eta}_1(0) = \eta_0, \bar{\eta}_2(0) = 0$, and $\dot{\bar{\eta}}_1(0) = \dot{\bar{\eta}}_2(0) = 0$

$$\bar{\eta}_1(t) = \frac{\eta_0}{2}\left(\cos(\omega_+ t) + \cos(\omega_- t)\right)$$

$$\bar{\eta}_2(t) = -\frac{\eta_0}{2\sqrt{\varepsilon}}\left(\cos(\omega_+ t) - \cos(\omega_- t)\right)$$



HW #10

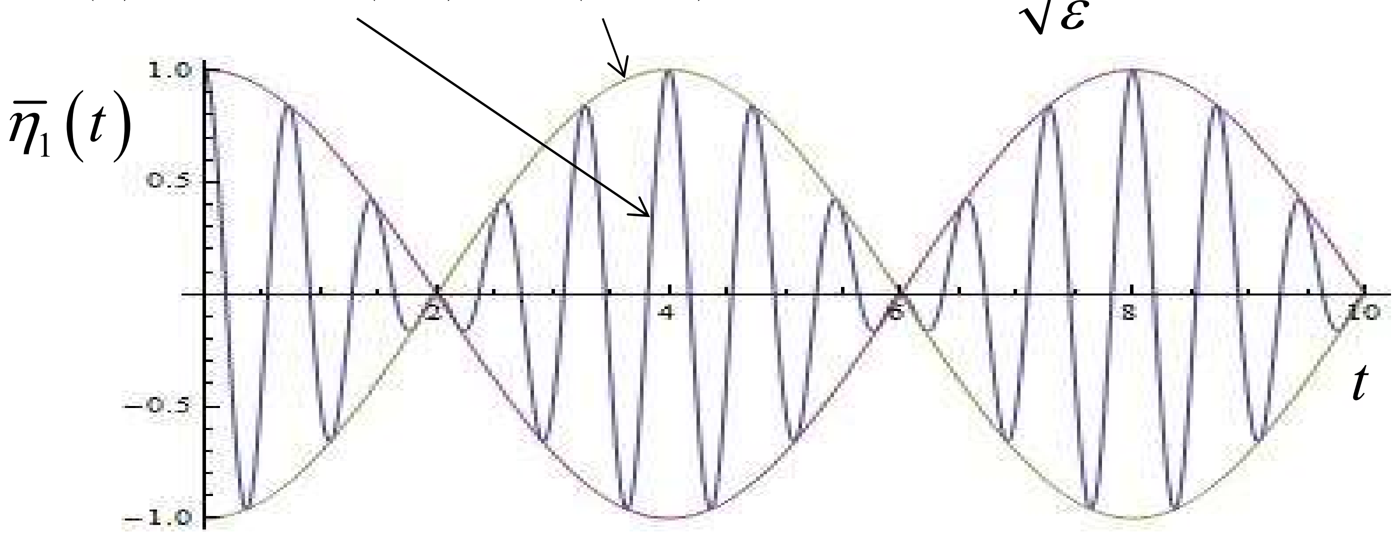
Using the following trig identities:

$$\cos\left(\omega_+ t\right)+\cos\left(\omega_+ t\right)=2\cos\left(\bar{\omega}t\right)\cos\left(\delta\omega\, t\right) \qquad \bar{\omega}=\frac{\omega_+ +\omega_-}{2}$$

$$-\cos\left(\omega_+ t\right)+\cos\left(\omega_+ t\right)=2\sin\left(\bar{\omega}t\right)\sin\left(\delta\omega\, t\right) \qquad \delta\omega=\frac{\omega_+ -\omega_-}{2}$$

Then, we can rewrite our solution in the following "beat" form:

$$\bar{\eta}_1\left(t\right)=\eta_0\cos\left(\bar{\omega}t\right)\cos\left(\delta\omega\, t\right) \qquad \bar{\eta}_2\left(t\right)=\frac{\eta_0}{\sqrt{\varepsilon}}\sin\left(\bar{\omega}t\right)\sin\left(\delta\omega\, t\right)$$



Review: Euler's Equations

$$
\begin{aligned}
I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 &= N_1 \\
I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 &= N_2 \\
I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 &= N_3
\end{aligned}
$$

(NOTE: all three equations have the same cyclic symmetry wrt the indices)

→ EOM describing the rigid body motion in the body axes

- all quantities must be expressed in the body coordinates

→ Body axes are chosen to align with the Principal Axes

-so that the Moments of Inertia Tensor is diagonalized and I's are the Principal Moments of Inertia

1st Example: Torque Free Motion of a Symmetric Top

A symmetric top means that: $I_1 = I_2 \neq I_3$

If $I_1 = I_2 > I_3$: the object will be a long cigar-like objects such as a juggling pin.

If $I_1 = I_2 < I_3$: the object will be a stubby objects such as a squashed pumpkin.

Euler equations are simplified in the torque free case:

$$I_1\dot{\omega}_1 = \left(I_2 - I_3\right)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = \left(I_3 - I_1\right)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = \left(I_1 - I_2\right)\omega_1\omega_2 = 0$$

$\vdots$

Torque Free Motion of a Symmetric Top

Nontrivial case ($\boldsymbol{\omega}$ is NOT along one of the principal axes):

$$\dot{\omega}_1 = -\Omega\omega_2$$
$$\dot{\omega}_2 = \Omega\omega_1$$
$$\boxed{\omega_3 = const}$$

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = const$$

With $\Omega^2 \geq 0$, we have the solution:

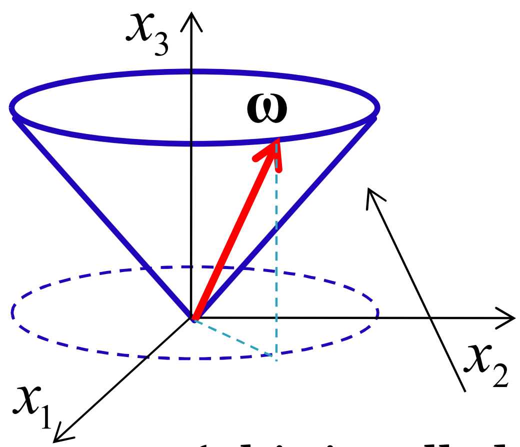
A, φ_0 will be determined by ICs

$$\boxed{\omega_1(t) = A\cos(\Omega t + \varphi_0) \quad \text{and} \quad \omega_2(t) = A\sin(\Omega t + \varphi_0)}$$

Torque Free Motion of a Symmetric Top

Geometric visualization:

In the "body" frame:



$\boldsymbol{\omega}$ rotates around x_3 with frequency

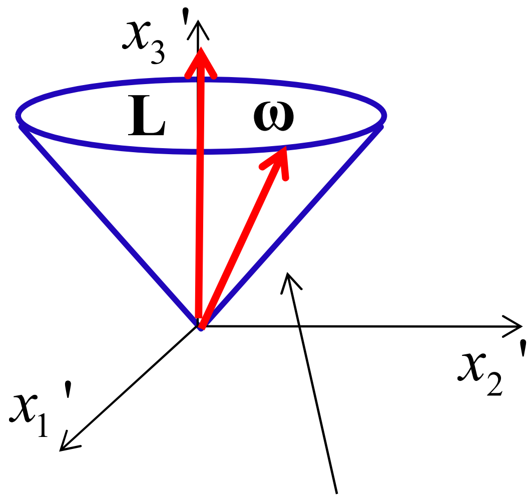
$$\Omega = \left(\frac{I_3 - I_1}{I_1} \right) \omega_3 = const$$

(This is called the "body" cone)

Torque Free Motion of a Symmetric Top

Geometric visualization:

In the "fixed" frame:



ω also rotates around $x_3{}'$ with frequency

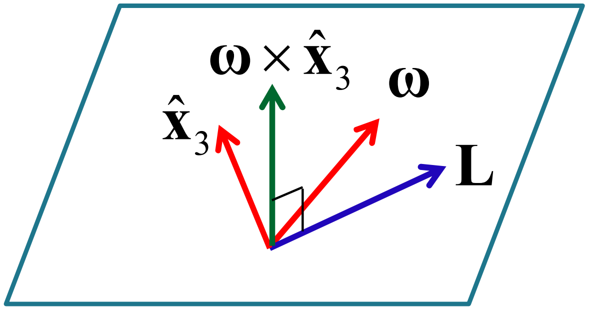
$$\Omega = \left(\frac{I_3 - I_1}{I_1} \right) \omega_3 = const$$

(This is called the "space" cone)

Torque Free Motion of a Symmetric Top

Observations (in the fixed axes) cont:

This means that all three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane.



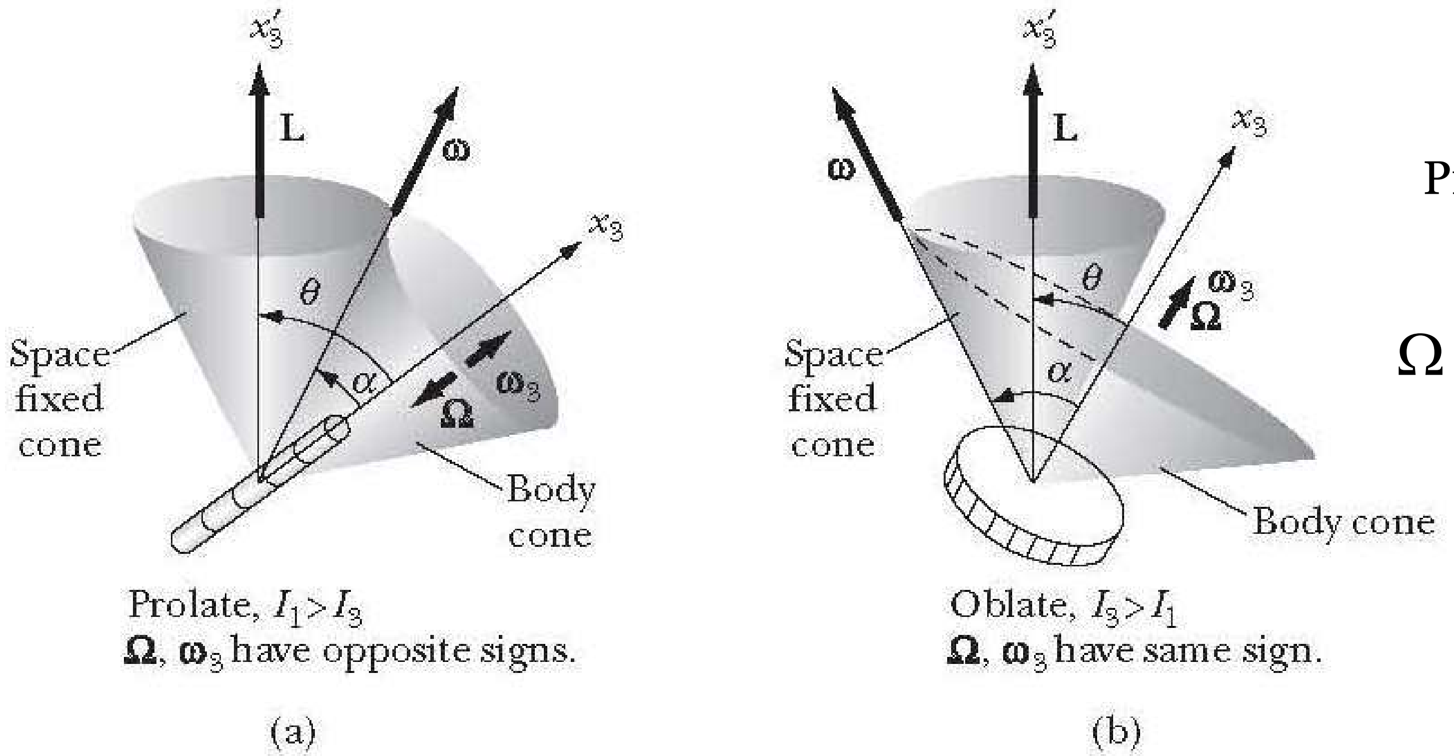
$$\mathbf{L} \cdot \left(\boldsymbol{\omega} \times \hat{\mathbf{x}}_3 \right) = 0$$ (for a symmetric top)

Summary:

- $\boldsymbol{\omega}$ precesses around the "body" cone
- $\boldsymbol{\omega}$ also precesses around the "space" cone
- All three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane
- $\mathbf{L}$ is chosen to align with $\hat{\mathbf{x}}_3{}'$ in the space axes

Torque Free Motion of a Symmetric Top

This can be visualized as the body cone rolling either inside or outside of the space cone !



Case 1: $I_1 > I_3$

Case 2: $I_1 < I_3$

Precession Rate

$$\Omega = \left(\frac{I_3 - I_1}{I_1} \right) \omega_3$$

Torque Free Motion of a Symmetric Top

Mathematica Animation:

http://demonstrations.wolfram.com/FreePrecessionOfARotatingRigidBody/

Stability of General Torque Free Motion

Consider torque-free motion for a rigid body with $I_1 > I_2 > I_3$

Again, we have chosen the body axes to align with the principal axes.

As an example, we will consider rotation near the x_1 axis (similar analysis can be done near the other two principal axes).

→ this means that we have,

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{x}}_1 + \lambda(t)\,\hat{\mathbf{x}}_2 + \mu(t)\,\hat{\mathbf{x}}_2$$

where $\lambda(t), \mu(t)$ are small time-dependent perturbation to the motion

For stability analysis, we wish to analyze the time evolution of these two quantities to see if they remain small or will they blow up.

Stability of General Torque Free Motion

The solution for the perturbations is oscillatory, i.e.,

$$\lambda(t) = Ae^{i\Omega t} + Be^{-i\Omega t}$$

and $$\mu(t) = A'e^{i\Omega t} + B'e^{-i\Omega t}$$ where *A, B, A', & B'* depends on ICs

$$\Omega^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^{\ 2} > 0$$

➡ Thus, both of the small perturbations are oscillatory and the rotation about the x_1 axis is stable !

Stability of General Torque Free Motion

With a similar calculation for rotation near the x_3 , one can show again that small perturbations are oscillatory and motion about the x_3 axis is stable.

However, a similar analysis will show that the oscillatory motion for the perturbations will become exponential if we consider rotation near the x_2 axis. (HW assignment)

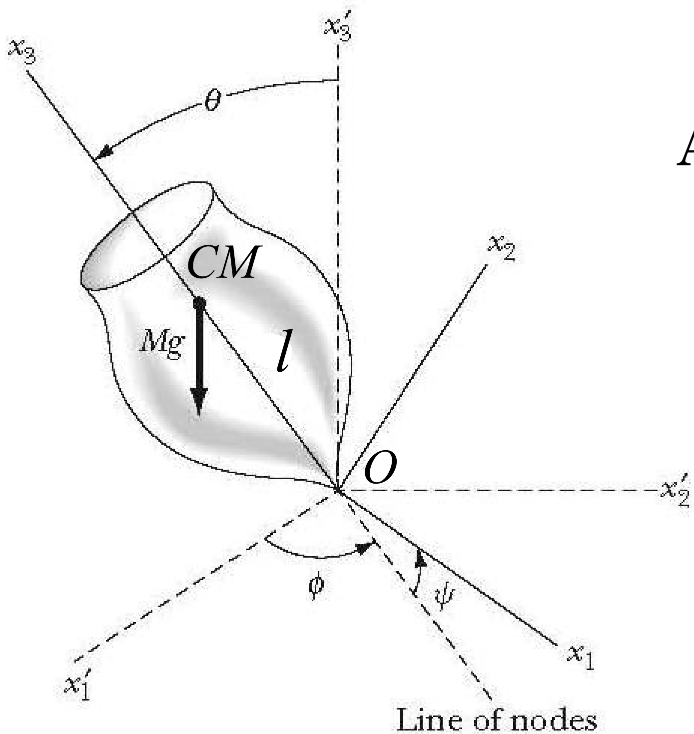
Summary:

Without any applied torque, motion around the principle axes with the largest and the smallest principal moments are stable while motion around the intermediate axis is unstable.

Symmetric Top in an Uniform Gravity Field

We have been looking at motion of torque-free rigid bodies.

Now, we consider a rigid body under the influence of gravity so that $U \neq 0$



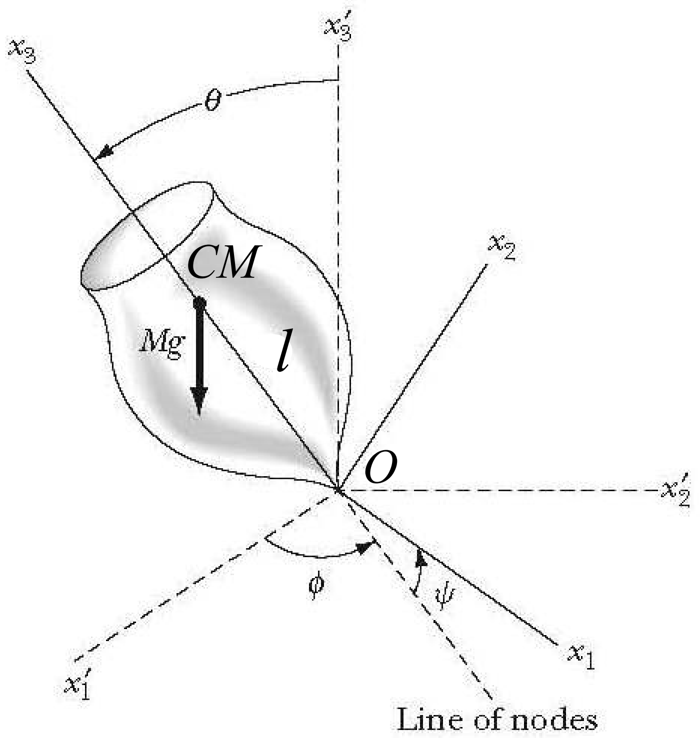
Assumptions:

- One point of the body remains fixed at the origin O but it not necessary coincides with the CM
- Again, we assume a symmetric top, i.e.,

$$I_1 = I_2 \neq I_3$$

Symmetric Top in an Uniform Gravity Field

To analyze the motion in the body frame, we can use the Euler's eqs:



$$I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = N_1$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 = N_2$$

$$I_3\dot{\omega}_3 = N_3 \quad \longleftarrow \quad \boxed{I_1 = I_2}$$

The Euler's equations provide a description for the time evolution of $(\omega_1, \omega_2, \omega_3)$ in the "body" axes

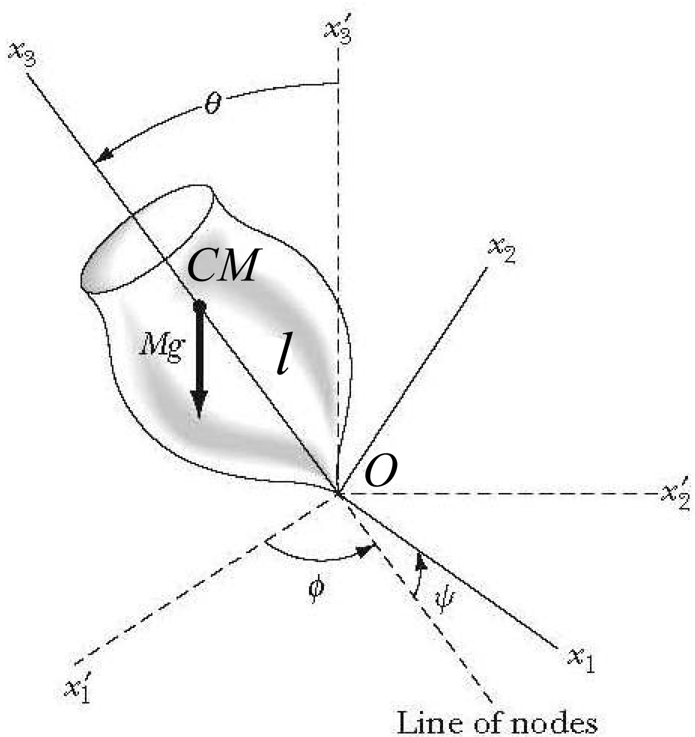
And, using $\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$, we can link the

description back to the Euler's angles.

Symmetric Top in an Uniform Gravity Field

Alternatively, we can use the Lagrangian method to directly obtain EOM for $\left(\dot{\phi},\dot{\theta},\dot{\psi}\right)$:

$$L = T - U = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta$$



Both ϕ,ψ are cyclic !

We immediately have the following two constants of motion:

$$\frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = p_\psi$$

$$\frac{\partial L}{\partial \dot{\phi}} = \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\cos\theta\dot{\psi} = p_\phi$$

Symmetric Top in an Uniform Gravity Field

Rescaling the two constants: $p_\phi = I_1 b \qquad p_\psi = I_1 a$

We can write ...

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \qquad\qquad \dot{\psi} = \frac{I_1}{I_3}a - \cos\theta\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)$$

Then, substituting $\dot{\phi}(\theta)$ and $\dot{\psi}(\theta)$ into the conservation of total energy equation,

$$E = T + U = \frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta$$

Rewriting, we then have the desired ODE for θ...

Symmetric Top in an Uniform Gravity Field

$$\frac{I_1}{2}\dot{\theta}^2 = E' - V_{eff}(\theta)$$

$$V_{eff}(\theta) = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

-The direct method is to integrate this to get $\theta(t)$. Then, substitute it back into the ODEs for $\dot{\phi}, \dot{\psi}$ and integrate to get $\phi(t), \psi(t)$.

Symmetric Top in an Uniform Gravity Field

- 3^{rd} Euler angle: $\dot{\psi}$ = spin about the body's symmetry axis
- 1st Euler angle: $\dot{\phi}$ = precession of the body's symmetry axis about the space $x_3{}'$ $(\hat{\mathbf{z}})$axis

(have seen in torque free case)

- 2^{nd} Euler angle: $\dot{\theta}$ = nutation (bobbing up & down) of the body symmetry axis (this is new)

Gyroscope demo !

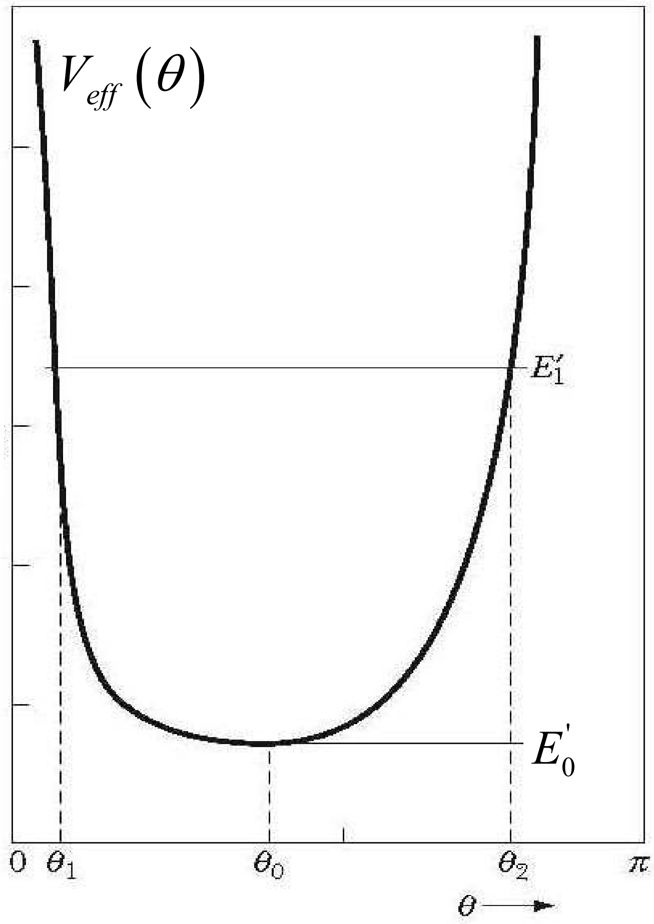
Symmetric Top in an Uniform Gravity Field

We can analyze the nutation by treating the $\dot{\theta}(t)$ equation as an effective potential problem,

$$\frac{I_1}{2}\dot{\theta}^2 = E' - V_{eff}(\theta)$$

$$V_{eff}(\theta) = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

Precession and Nutation for a Symmetric Top



Overview:

1. There is a minimum value of

$$E' = E_0' = V_{eff}(\theta_0)$$

for which there is only ONE allowed value for $\theta = \theta_0$ (pure precession).

2. For larger values of $E' > E_0'$ such as

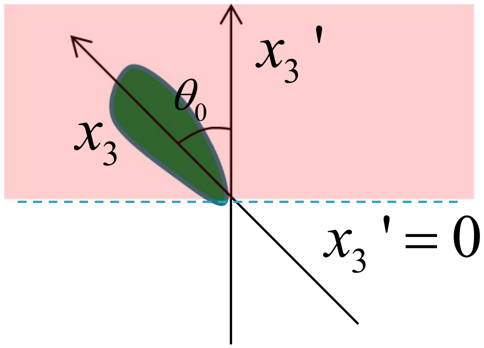
$$E' = E_1'$$

θ is bounded between 2 values:

$$\theta_1 \le \theta \le \theta_2$$

This is the case of nutation. We will look at these two situations closer next.

Precession for a Symmetric Top



Two subcases:

Case 1a: $\theta_0 < \pi/2$ The tip of the top is above the horizontal plane ($x_3\,' = 0$ in the fixed frame).

For a solution of steady precession at a fixed tilt $\theta_0 < \pi/2$

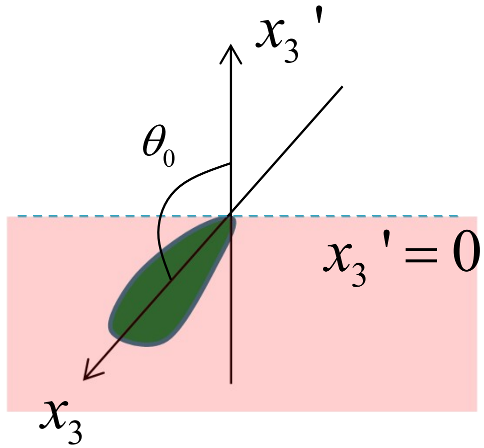
$$\omega_3 \geq \frac{2}{I_3}\sqrt{I_1 M g l \cos\theta_0} = \omega^*$$

So, ω_3 must be fast enough, i.e., $\omega_3 \gg \omega^*$(fast top)

$$\dot{\phi}_{0\mp} = \begin{cases} \dfrac{Mgl}{I_3\omega_3} & \text{Slow precession} \\[2ex] \dfrac{I_3\omega_3}{I_1\cos\theta_0} & \text{Fast precession} \end{cases}$$

(note: the fast precession is independent of gravity g)

Precession for a Symmetric Top



Case 1b: $\theta_0 > \pi/2$ The tip of the top is below the horizontal plane ($x_3' = 0$ in the fixed frame). Top is supported by a point support (show).

➡ No special condition on ω_3 .

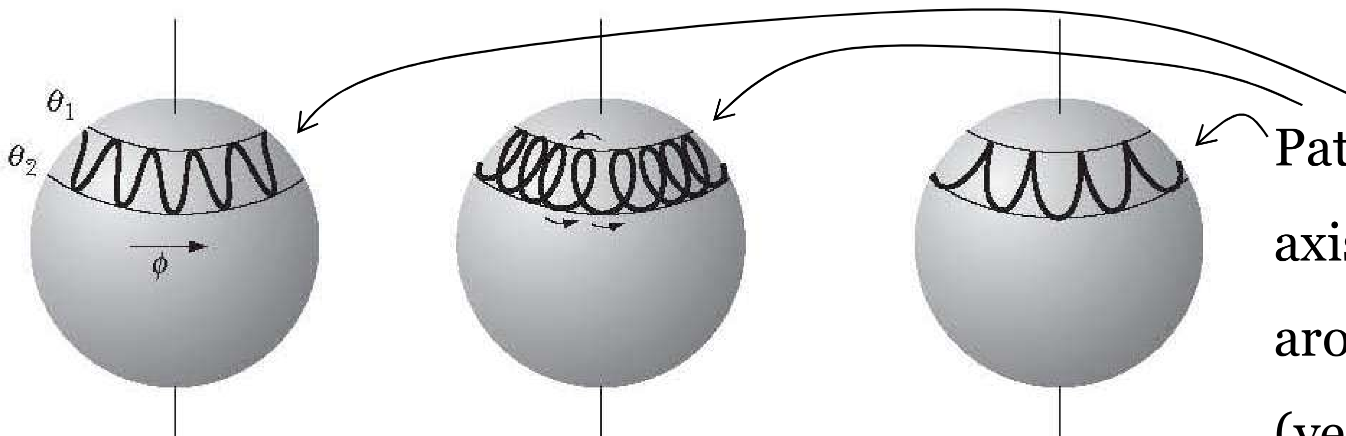
With top started with initial condition $\theta_0 > \pi/2$, it will remain below the horizontal plane and precesses around the fixed axis x_3' .

Nutation for a Symmetric Top

Case 2: $\theta_1 \le \theta \le \theta_2$ General situation with $E' > E_0' = V_{eff}(\theta_0)$. The body axis x_3 will blob up and down as it precesses around the fixed axis x_3' (nutation).

The precession rate of the body axis x_3 is described by: $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta}$

So, depending on *a* and *b*, $\dot{\phi}$ might or might not change sign... and we have the following three cases: (*a* and *b* are proportional to the 2 consts of motion: p_ψ, p_ϕ)



$b > a$

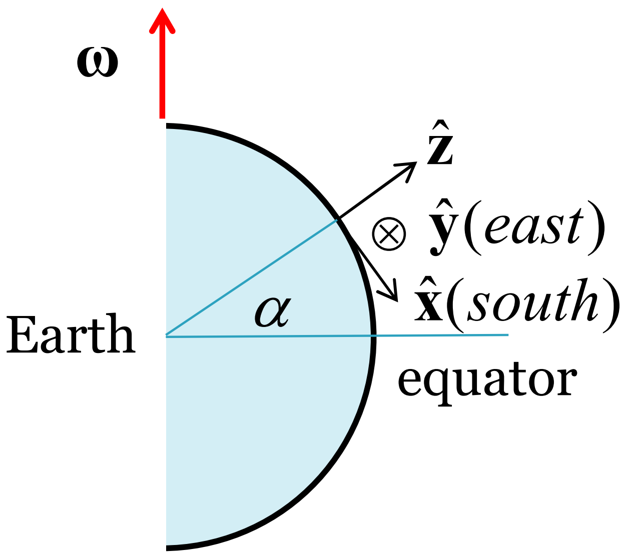
$b < a$
(reversal possible)

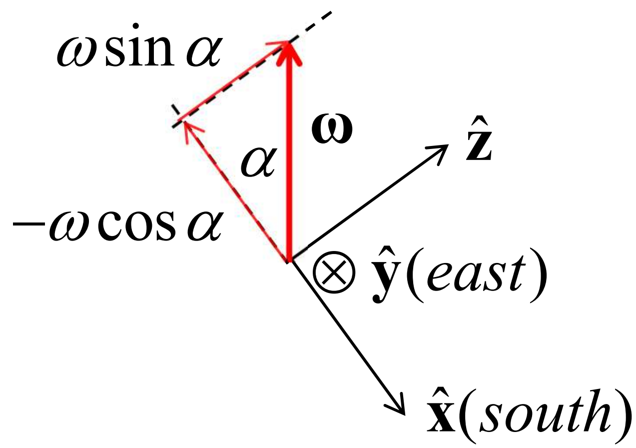
$b = a\cos\theta_1$
$\dot{\phi}(\theta_1) = 0$

Paths trace by the body axis x_3 as it precesses around the fixed axis x_3' (vertical).

Prob 4.21

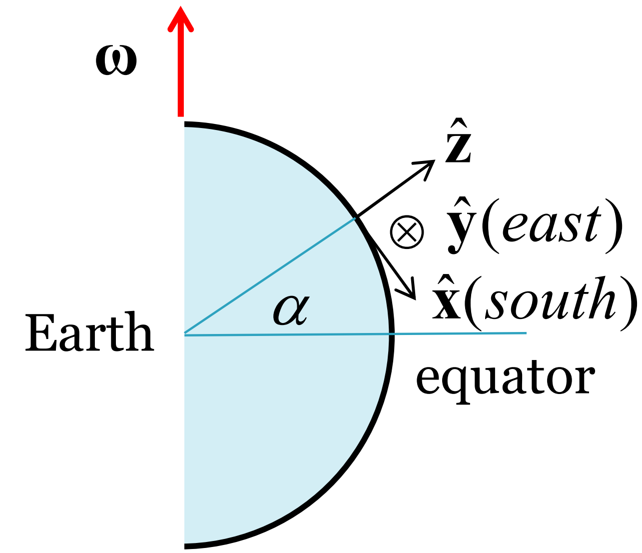
Here is the geometry of the problem:





$$\boldsymbol{\omega} = -\omega \cos\alpha\ \hat{\mathbf{x}} + \omega \sin\alpha\ \hat{\mathbf{z}}$$

Prob 4.21



We will consider the Coriolis force only (no centrifugal force), the equation of motion for a particle moving with a velocity $\mathbf{v}$ *in the rotating frame* is,

$$\boldsymbol{\omega} = -\omega\cos\alpha\;\hat{\mathbf{x}} + \omega\sin\alpha\;\hat{\mathbf{z}}$$

$$\mathbf{F}_{eff} = m\mathbf{a}_r$$

$$-2m(\boldsymbol{\omega}\times\mathbf{v}) - mg\hat{\mathbf{z}} = m\mathbf{a}_r$$

$$\Rightarrow \boxed{\dot{\mathbf{v}} = -2(\boldsymbol{\omega}\times\mathbf{v}) - g\hat{\mathbf{z}}}$$

$$\boldsymbol{\omega}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\omega\cos\alpha & 0 & \omega\sin\alpha \\ v_x & v_y & v_z \end{vmatrix}$$

$$\boldsymbol{\omega}\times\mathbf{v} = -v_y\omega\sin\alpha\;\hat{\mathbf{x}} + \left(v_x\omega\sin\alpha - v_z\omega\cos\alpha\right)\hat{\mathbf{y}} + v_y\omega\cos\alpha\;\hat{\mathbf{z}}$$

Prob 4.21

In component form, the equation of motion is:

$$\begin{cases} \dot{v}_x = 2\omega \sin\alpha \, v_y \\ \dot{v}_y = -2\omega\left(\cos\alpha \, v_z + \sin\alpha \, v_y\right) \\ \dot{v}_z = 2\omega \cos\alpha \, v_y - g \end{cases}$$

For ω small, we can try to solve this coupled differential equation perturbatively…

0th – order in ω (i.e., take $\omega = 0$), we have
$$\begin{cases} \dot{v}_x^{(0)} = 0 \\ \dot{v}_y^{(0)} = 0 \\ \dot{v}_z^{(0)} = -g \end{cases}$$
(particle in free fall)

Prob 4.21

For an initial condition $\mathbf{v}(0) = v_{z0}\ \hat{\mathbf{z}}$, the 0th order solution has the general solution:

$$\begin{cases} v_x^{(0)}(t) = 0 \\ v_y^{(0)}(t) = 0 \\ v_z^{(0)}(t) = v_{z0} - gt \end{cases} \quad \text{and} \quad z^{(0)}(t) = z_0 + v_{z0}t - \frac{1}{2}gt^2$$

Now, plug in the 0th – order solution back into the ODE and the resulting ODE is 1st order in ω, $O(\omega)$,

1st – order:

$$\begin{cases} \dot{v}_x^{(1)} = 2\omega \sin\alpha\ v_y^{(0)} & = 0 \\ \dot{v}_y^{(1)} = -2\omega\left(\cos\alpha\ v_z^{(0)} + \sin\alpha\ v_y^{(0)}\right) & = -2\omega\cos\alpha\left(v_{z0} - gt\right) \\ \dot{v}_z^{(1)} = 2\omega\cos\alpha\ v_y^{(0)} - g & = -g \end{cases}$$

Prob 4.21

So, up to 1st order in ω, the Coriolis effect only affects the motion in the y direction.

$$\begin{cases} \dot{v}_x^{(1)} = 0 \\ \dot{v}_y^{(1)} = -2\omega\cos\alpha\left(v_{z0} - gt\right) \\ \dot{v}_z^{(1)} = -g \end{cases}$$

The deflection in the y-direction (to the lowest order in ω) is then given by:

$$\boxed{\dot{v}_y = -2\omega\cos\alpha\left(v_{z0} - gt\right)}$$

Prob 4.21

$$\dot{v}_y = -2\omega\cos\alpha\left(v_{z0} - gt\right)$$

For the initial condition: $\mathbf{v}(0) = v_{z0}\,\hat{\mathbf{z}}$

$$\Rightarrow \quad v_y(t) = -2\omega\cos\alpha\, v_{z0}t + g\omega\cos\alpha\, t^2 \qquad \left[v_y(0) = 0\right]$$

$$\boxed{y(t) = -\omega\cos\alpha\, v_{z0}t^2 + \frac{1}{3}g\omega\cos\alpha\, t^3} \qquad \left[y(0) = 0\right]$$

Now, we consider the two different cases:

Case 1: (Straight up $v_{z0} = v_0 > 0$ from ground $z(0) = 0$):

We still have the same free fall motion in the z-direction,

$$v_z(t) = v_0 - gt \quad \text{and} \quad z(t) = v_0 t - \frac{1}{2}gt^2 \quad \left[z(0) = 0\right]$$

Prob 4.21

Thus, the particle will slow down as it goes up and reaches the top of its trajectory at time $t = v_0/g$ and at a height of

$$z\left(\frac{v_0}{g}\right) = h = \frac{v_0^2}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}$$

Since up-and-down is time symmetric, it will take $t = 2v_0/g$ for the particle to go up and come back down.

Prob 4.21

The accumulated y-deflection will be:

$$y\left(\frac{2v_0}{g}\right) = -\omega\cos\alpha\ v_0\left(\frac{2v_0}{g}\right)^2 + \frac{1}{3}g\omega\cos\alpha\left(\frac{2v_0}{g}\right)^3$$

$$= \omega\cos\alpha\ \frac{v_0^3}{g^2}\left(-4+\frac{8}{3}\right) = -\frac{4}{3}\omega\cos\alpha\ \frac{v_0^3}{g^2}$$

("-" to the west)

Prob 4.21

Case 2: (*Dropping* from the same height reached as in Case 1):

$$z(0) = h = \frac{v_0^2}{2g} \qquad v(0) = 0$$

The time it take for the particle to reach the ground from a height of h is,

$$z(t) = h + \cancel{v_{z0}} t - \frac{1}{2} g t^2 = 0 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}} = \frac{v_0}{g}$$

Prob 4.21

The accumulated y-deflection will then be:

$$y(t) = -\omega\cos\alpha\, v_{z0} t^2 + \frac{1}{3} g\omega\cos\alpha\, t^3$$

$v_{z0} = 0$ since it is being dropped

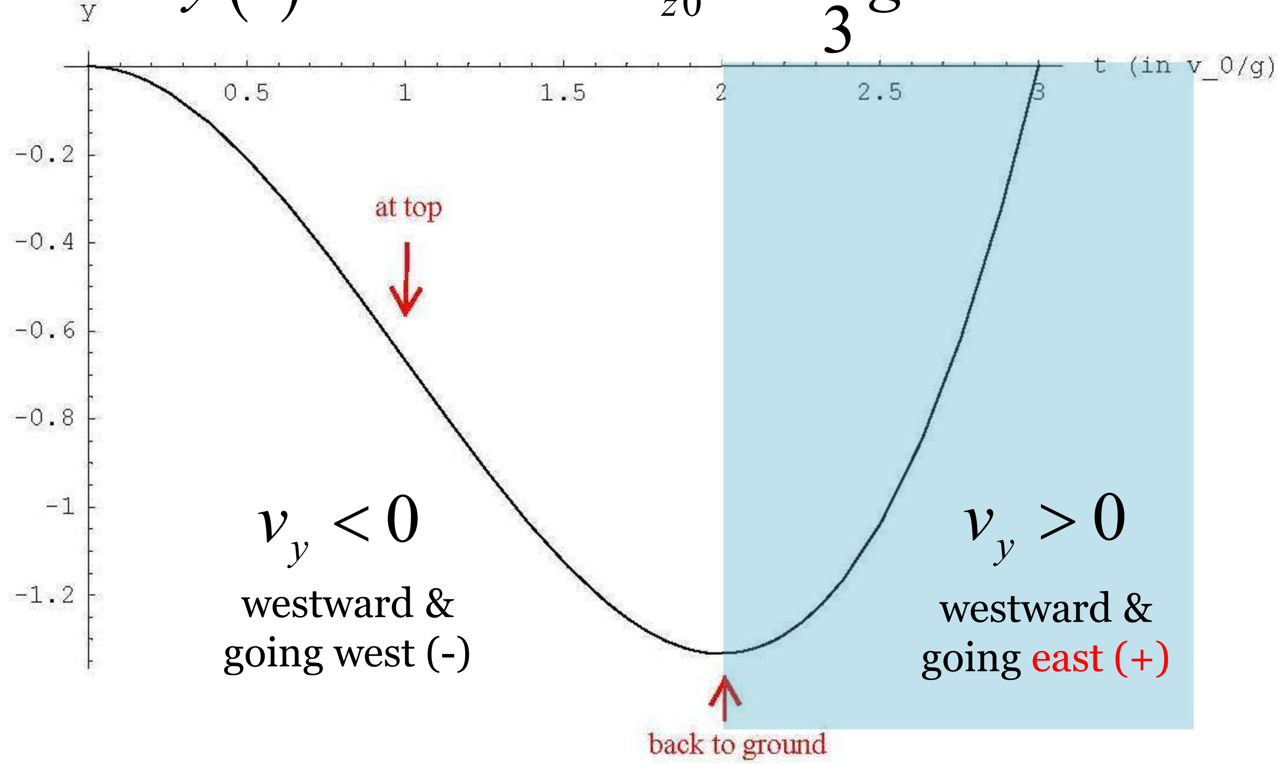
$$y\left(\frac{v_0}{g}\right) = \frac{1}{3} g\omega\cos\alpha \left(\frac{v_0}{g}\right)^3 = +\frac{1}{3}\omega\cos\alpha\,\frac{v_0^3}{g^2}$$

("+" to the east)

$$\Rightarrow \quad y_{case1} = -4y_{case2}$$

Prob 4.21

$$y(t) = -\omega\cos\alpha\, v_{z0} t^2 + \frac{1}{3} g\omega\cos\alpha\, t^3$$



Note: There are two competing terms here. The negative term dominates initially but the positive term makes $y(t)$ + when $t \geq \frac{3v_0}{g}$

As the particle goes up, it accumulated so much westward deflection, it takes some time for it to make up and be eastward as it goes down. Dropping from $z=h$ will start deflecting eastward right away.

HW #11

Pay attention to the ordering of indices and also which one is free and which one is being summed over

$$\left[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})\right] = \varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})_j(\mathbf{C}\times\mathbf{D})_k$$

$$= \varepsilon_{ijk}\left(\varepsilon_{jmn}a_m b_n\right)\left(\varepsilon_{kpq}c_p d_q\right)$$

$$= \varepsilon_{ijk}\varepsilon_{jmn}\varepsilon_{kpq}a_m b_n c_p d_q \qquad \text{[they are just \# now]}$$

$$= \varepsilon_{jmn}\left(\varepsilon_{kij}\varepsilon_{kpq}\right)a_m b_n c_p d_q \qquad \left[\varepsilon_{ijk} = \varepsilon_{kij}\right]$$

$$= \varepsilon_{jmn}\left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right)a_m b_n c_p d_q$$

$$= \varepsilon_{jmn}\left(a_m b_n c_i d_j - a_m b_n c_j d_i\right) \qquad \text{[collapsing } \delta_{ij}\text{]}$$

$$= \left(\left(\varepsilon_{jmn}a_m b_n\right)d_j\right)c_i - \left(\left(\varepsilon_{jmn}a_m b_n\right)c_j\right)d_i$$

$$= \left[(\mathbf{A}\times\mathbf{B})\cdot\mathbf{D}\right]\mathbf{C} - \left[(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}\right]\mathbf{D}$$